Transforming Graphs, Exponential and Logarithm Functions

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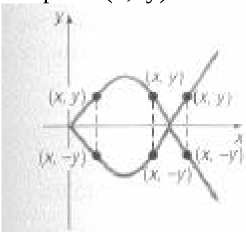
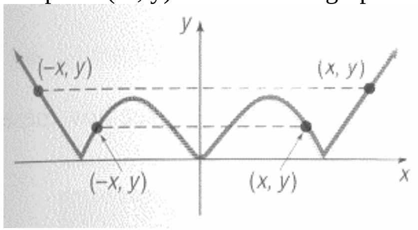
Symmetry with respect to x-axis

- A graph is said to be **Symmetric with respect to the x-axis** if, for every point (x, y) on the graph, the point (x, -y) is also on the graph.



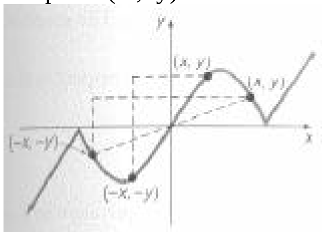
Symmetry with respect to y-axis

- A graph is said to be **Symmetric with respect to the y-axis** if, for every point (x, y) on the graph, the point (-x, y) is also on the graph.



Symmetry with respect to origin

- A graph is said to be **Symmetric with respect to the origin** if, for every point (x, y) on the graph, the point (-x, -y) is also on the graph.



Reflection

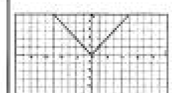
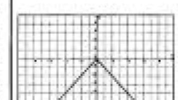
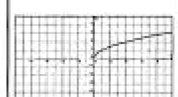
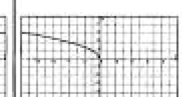
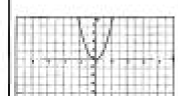
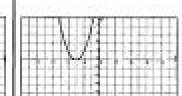
- The graph of $y = -f(x)$ is a reflection in the x-axis of the graph of $y = f(x)$.

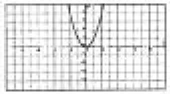
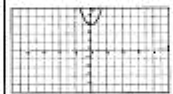
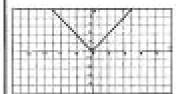
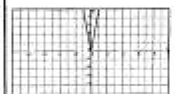
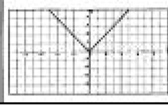
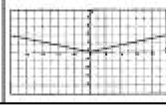
Translation

- Translating Upward or Downward
  - If $k > 0$, then the graph of $y = f(x) + k$ is an **Upward Translation** of the graph of $y = f(x)$ and the graph of $y = f(x) - k$ is a **Downward Translation** of the graph of $y = f(x)$
- Translating to the Right and Left
  - If $h > 0$, then the graph $y = f(x - h)$ is a **Translation to the Right** of the graph of $y = f(x)$, and the graph of $y = f(x + h)$ is a **Translation to the Left** of the graph of $y = f(x)$

Stretching and Shrinking

- If $a > 1$, then the graph of $y = af(x)$ is obtained by **Stretching** the graph of $y = f(x)$.
- If $0 < a < 1$, then the graph of $y = af(x)$ is obtained by **Shrinking** the graph of $y = f(x)$.

Function	Description of change	Example	
		original	*transformed*
$y = -f(x)$	reflection across the *x*-axis	$y = \|x\|$ 	$y = -\|x\|$ 
$y = f(-x)$	reflection across the *y*-axis	$y = \sqrt{x}$ 	$y = \sqrt{-x}$ 
$y = f(x + h)$	translation of $-h$ in the *x*-direction	$y = x^2$ 	$y = (x + 3)^2$ 

Function	Description of change	Example	
		original	*transformed*
$y = f(x) + h$	translation of $+h$ in the y-direction	$y = x^2$ 	$y = x^2 + 3$ 
$y = a \cdot f(x)$ $a > 1$	stretching by factor of a in the y-direction	$y = \lvert x \rvert$ 	$y = 5\lvert x \rvert$ 
$y = a \cdot f(x)$ $0 < a < 1$	shrinking by factor of a in the y-direction	$y = \lvert x \rvert$ 	$y = \frac{1}{5}\lvert x \rvert$ 

Example

Sketch the graph of $y = \sqrt{x+2}$.

Example

Sketch the graph of $y = -2\,|\,x\,| + 3$.

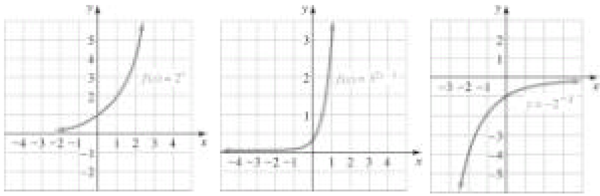
Example

Sketch the graph of $y = x^2 + 2x + 3$.

Exponential Functions

- An Exponential Function is a function of the form
  - $f(x) = a^x$

  where $a > 0$ and $a \neq 1$



Graphing Exponential Function

- The graph of the function $f(x) = a^x$
  - Pass through the point (0, 1).
  - Has domain $(-\infty, \infty)$ and range $(0, \infty)$.
  - Approaches, but does not touch, the x-axis.
  - Increase from Left to Right if $a > 1$.
  - Decrease from Left to Right if $0 < a < 1$.

One-to-One Property of Exponential Functions

- For $a > 0$ and $a \neq 1$,
  - If $a^m = a^n$, then $m = n$

Example

- Sketch the graph of $f(x) = 2^{x-1} - 2$

Example

Solve $25^x = \frac{1}{5^{3x-1}}$.

Logarithm Functions

- For a > 0 and a ≠ 1,
  - $y = \log_a x$ if and only if $a^y = x$.

Graphing Logarithm Function

- The graph of the function $f(x) = \log_a x$
  - Pass through the point (1, 0).
  - Has domain (0, ∞) and range (–∞, ∞).
  - Approaches, but does not touch, the y-axis.
  - Increase from Left to Right if a > 1.
  - Decrease from Left to Right if 0 < a < 1.

Example

- Sketch the graph of $g(x) = \log_2 (x)$ and compare it to the graph $y = 2^x$.

Example

- Evaluate $\log_5 (625)$

Properties of Logarithms

Product Rule for Logarithms

$$\log_a(MN) = \log_a(M) + \log_a(N)$$

Quotient Rule for Logarithms

$$\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$$

Power Rule for Logarithms

$$\log_a(M^N) = N \cdot \log_a(M)$$

Summary of Properties

Properties of Logarithms

If M, N, and a are positive numbers, $a \neq 1$, then

1. $\log_a(a) = 1$ **2.** $\log_a(1) = 0$

3. $\log_a(a^M) = M$ Inverse properties **4.** $a^{\log_a(M)} = M$

5. $\log_a(MN) = \log_a(M) + \log_a(N)$ Product rule

6. $\log_a\left(\frac{M}{N}\right) = \log_a(M) - \log_a(N)$ Quotient rule

7. $\log_a\left(\frac{1}{N}\right) = -\log_a(N)$ **8.** $\log_a(M^N) = N \cdot \log_a(M)$ Power rule

Changing the Base

Base-Change Formula

If a and b are positive numbers not equal to 1 and M is positive, then

$$\log_a(M) = \frac{\log_b(M)}{\log_b(a)}.$$

- Example:
  - Evaluate $\log_7 99$ using base-change formula

Strategy for Solving Equations

Solving Exponential and Logarithmic Equations

1. If the equation has a single logarithm or a single exponential expression, rewrite the equation using the definition $y = \log_a(x)$ if and only if $a^y = x$.
2. Use the properties of logarithms to combine logarithms as much as possible.
3. Use the one-to-one properties:
   a) If $\log_a(m) = \log_a(n)$, then $m = n$.
   b) If $a^m = a^n$, then $m = n$.
4. To get an approximate solution of an exponential equation, take the common or natural logarithm of each side of the equation.

Example

- Solve $3\log_8 (2x - 1) = 4$.

Example

- Solve $4^{2x} = 3^{3x-1}$.

References

- Algebra for College Students (Ch. 10 – 11)